

# **Analytical Model of Route-Based Pricing for Time Dependent Traffic Assignment**

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### **Abstract**

Following basic models in time dependent traffic assignment, we propose a flexible route-based pricing scheme where the tolls change with a given block of departure time. We transform the usual bi-level road pricing problem to a single-level problem via variational inequality and Karush-Kuhn-Tucker conditions. With input from time dependent origin-destination (OD) matrix, we study a fixed demand model: fixing the total demand over the study horizon, but allowing it to vary within the entire modelling horizon. Our aim is to add some basic features of dynamic traffic models to static traffic assignment in order to derive a flexible and time dependent route-based optimal tolling scheme that can be used to efficiently distribute traffic in and over time in a network. Incorporating Small's idea, the optimal route-based toll further takes into account the cost involved in shifting departure time of a user from one time interval to another.

## **Keywords**

Time dependent traffic models, Time dependent user equilibrium, Variational inequality, Time dependent path-based tolls, Transfer costs.

# 1 INTRODUCTION

Over the recent years researchers have turned attention to modeling dynamic traffic assignment (DTA) owing to the fact that the so called static traffic assignment (STA) does not represent the real traffic dynamics. In literature, only small has been done to analytically determine the dynamic tolls corresponding to DTA. Most works are based on simulation of the link tolls which tries to update the tolls in each iteration given the objective value [7, 8]. A dynamic congestion pricing scheme using a game-theoretic evolutionary learning model was developed in [5]. Their model recognizes several important behavioral features related to the response of users to the congestion pricing strategy. Using optimal control method, [4] developed dynamic congestion pricing models for general traffic networks. Some drawbacks of optimal control technique are mentioned in [3]. Byung-Wook [12] considers the problem of dynamic congestion pricing that determines optimal time-varying tolls for a prespecified subset of arcs with bottleneck on a congested general traffic network. He formulated the problem as a two-person nonzero-sum dynamic Stackelberg game model analysing the characteristics of the Stackelberg equilibrium solution. His work is of course too specific. It is important to note that there has been no generally accepted method for modelling DTA, but there have been accepted features that a typical DTA model should have. In this paper, we follow the DTA models as specified in [10, 3, 6]. The underlying conditions specified in [10, 3, 6] has been used widely in modelling DTA.

In most models of DTA, the lower level problem also known as the user problem has always been transformed into a variational inequality VI, the VI will then serve as an equilibrium condition. The problem of optimal road pricing thus becomes a search for a pricing scheme that optimizes certain objective subject to equilibrium constraint. This problem is often refers to as mathematical program with equilibrium constraint (MPEC). It is generally known that MPECs are hard to solve. To this we propose for a time dependent traffic assignment (TDTA) a transformed problem that requires a set of linear constraints to ensure existence of time dependent user equilibrium (TDUE) in the lower level. The bi-level problem can then be transformed to a single level program. The problem can be solved in two stages, first we solve for system optimal route-flow pattern, and then using the transformed TDUE linear conditions, we search for an optimal route-based tolls that can induce the optimal route-flow in the network. Such models already exist for link-based tolls in static traffic assignment [13, 9]. In this paper we extend their work to TDTA models. It is important to make clear that we are not trying to analytically determine route tolls for fully dynamic traffic assignment (DTA) that captures all traffic dynamics, instead, we will utilize some features of DTA and formulate a time dependent model. Since all fully DTA models are simulation based, we cannot do any analytical exercise with such models. We use chain of time blocks to replace the almost continuous time model in the DTA models. In fact, we are incorporating basic dynamic features in the STA. With the time dependent traffic assignment (TDTA) model, we design a tolling scheme wherein the tolls change with each block of departure time interval. For our model, we take a fixed demand which is read from OD matrices built by observing traffic over time. With the input from the matrix, traditionally, fixed demand models for static traffic assignment (STA) fixes the demand for each time interval as read from the OD matrix and searches for optimal flow pattern for this time slot. In our model, we do not fix the demand for each discretized time interval, we allow the demand to be 'elastic' within the modelling horizon, but still ensure that the total amount of traffic over the entire modelling horizon (as read from the input matrix) is realized. We allow for this flexibility in demand because counted traffic (user behaviour - or user equilibrium) may be far from system optimal traffic flow as we will see later. In this way, demands are efficiently distributed over time. Furthermore, we have include in our model the fact that observed departure travel pattern is not necessarily preferred departure pattern for the road users. To check for this, we have borrowed the idea of Small [11] and that of Eric and Martie [2] to account for the cost involved in shifting departure time of a user from a given time slot to another.

To implement the route-based tolls, it may be needed that cars are equipped with tracking system, or at least, there should be a system that determines/tracks the route followed by a car for a given origin destination.

## 2 NOTATION AND FEASIBILITY CONDITIONS

Let  $G = \{N, A\}$  denote a transportation network consisting of a set of nodes  $N$  and a set of links  $A$ . Let  $P$  be the set of all routes in  $G$  and  $p \in P$  the index for routes with  $G$ . One or more routes  $p \in P$  may exist between origin( $r$ )-destination( $s$ ) pair  $rs = w \in W$ . We use  $W$  to denote the set of all origin-destination (OD) pairs. Every route  $p$  is comprised of one or more links  $a \in A$ . We use a discrete time formulation in which the whole studied time period  $T$  is divided into a certain number of small time intervals, denoted by  $t^i$  [7]. These discrete time intervals  $t^i$  with  $i = 1, 2, \dots, T$  are such that they correspond to the departure times. For example, if the study time period is the from 6:00hrs to 12:00hrs, then the departure time intervals  $t^i$  can be  $t^1 = 6 : 00hrs - 7 : 00hrs$ ,  $t^2 = 7 : 00hrs - 8 : 00hrs$ ,  $t^3 = 8 : 00hrs - 9 : 00hrs$ , and so on. Note that this length of the interval is arbitrarily chosen. We will consider one user class model, heterogeneous users model is straightforward. Next we give the rest of the notations used in this paper. Our aim is to analytical determine the optimal path toll  $\theta_p^w(t^i)$  to be paid on

route  $p \in P_w$  when a user travelling to destination  $s$  departs origin  $r$  during the time interval  $t^i$ .

**TABLE 1 Notations**

$x_a(t^i)$	number of vehicles traversing link $a$ during time $t^i$
$x_{ap}^w(t^i)$	number of vehicles traversing link $a$ and route $p$ between OD pair $w$ during departure time interval $t^i$
$u_a(t^i)$	inflow rate of link $a$ during departure time interval $t^i$
$u_{ap}^w(t^i)$	inflow rate of link $a$ on route $p$ between OD pair $w$ during departure time interval $t^i$
$v_a(t^i)$	exit flow rate of link $a$ during departure time interval $t^i$
$v_{ap}^w(t^i)$	exit rate of link $a$ on route $p$ between $w^{th}$ OD pair
$E_p^w(t^i)$	cummulative number of vehicles arriving destination $s$ from origin $r$ on route $p$ by time $t^i$
$e_p^w(t^i)$	arrival flow rate at the destination for the $w^{th}$ OD pair on route $p$ during departure time interval $t$
$f^w(t^i)$	departure flow rate for the $w^{th}$ OD pair during time $t^i$
$f_p^w(t^i)$	departure flow rate into route $p$ for the $w^{th}$ OD pair during time $t^i$
$F$	set of all feasible path flows
$P_w$	set of all paths belonging to the $w^{th}$ OD pair
$A(n)$	set of links whose tail node is $n$
$B(n)$	set of links whose head node is $n$
$\tau_a(t^i)$	travel time over link $a$ for flows entering link $a$ during departure time interval $t^i$
$\eta_p^w(t^i)$	travel time experienced over route $p$ by users of the $w^{th}$ OD pair during departure time interval $t^i$
$\hat{d}^w(t^i)$	demand for the $w^{th}$ OD pair during time $t^i$ as observed from the input OD matrix. It is the number of travellers departing origin $r$ during time $t^i$ towards destination $s$
$\bar{d}^w(t^i)$	optimized demand for the $w^{th}$ OD pair at time $t$ . It is the optimize number of travellers departing origin $r$ during time $t^i$ towards destination $s$
$c_{t^i t^j}^w$	cost involved in shifting departure time of a user from $t^i$ to $t^j$ for $w^{th}$ OD pair
$y_{t^i t^j}^w$	number of users that were departing during $t^i$ but are rescheduled to depart during $t^j$ for the $w^{th}$ OD pair
$z_{t^i t^j}^w$	number of users who prefer departing during $t^i$ but are actually during $t^j$ for the $w^{th}$ OD pair

Flow conservation constraints

$$f_p^w(t^i) = \sum_{a \in A(r)} u_{ap}^w(t^i) \quad \forall i, w, p \in P_w, \quad [\alpha] \quad (1)$$

$$e^w(t^i) = \sum_{a \in B(s)} \sum_p v_{ap}^w(t^i) \quad \forall i, w \quad (2)$$

$$\sum_{a \in A(n)} u_{ap}^w(t^i) = \sum_{a \in B(n)} v_{ap}^w(t^i) \quad \forall i, w, p \in P_w, n \quad [\gamma] \quad (3)$$

$$\sum_p f_p^w(t^i) = d^w(t^i) \quad \forall i, w \quad [\delta] \quad (4)$$

$$\sum_j y_{t^j t^i}^w = d^w(t^i) \quad \forall j, w \quad [\delta] \quad (5)$$

$$\sum_i y_{t^j t^i}^w = \hat{d}^w(t^j) \quad \forall i, w \quad [\zeta] \quad (6)$$

$$\sum_i d^w(t^i) = \sum_i \hat{d}^w(t^i) \quad \forall w \quad [\zeta] \quad (7)$$

$d^w(t^i)$  is the optimizable demand for the  $w^{th}$  OD pair during the  $i^{th}$  departure time interval.  $\hat{d}^w(t^i)$  is the observed demand patterns (as read from the input time-dependent OD matrix) for the  $w^{th}$  OD pair during the  $i^{th}$  departure time interval. Note that since our model focuses on optimal route tolls  $\theta_p^w(t^i)$  that induce optimal route flow  $f_p^w(t^i)$  during departure time  $t^i$ , (2) can as well be omitted since Eqn (4) takes care of arrival flows. The Greek letters in the square brackets are the Karush-Kuhn-Tucker (KKT) multipliers associated with the constraints.

Definitional constraints

for all  $i$ , the following conditions hold:

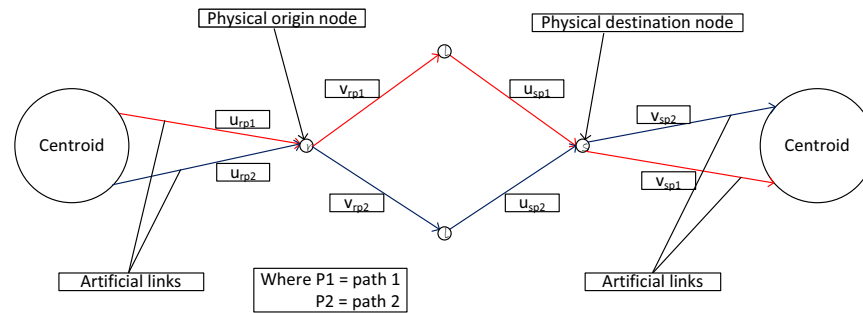
$$\sum_{w,p} u_{ap}^w(t^i) = u_a(t^i), \quad \sum_{w,p} v_{ap}^w(t^i) = v_a(t^i) \quad \forall a \quad (8)$$

$$u_{bp}^w(t^i) = v_{ap}^w(t^i) \quad \forall a \in A(r), A(s), b \in B(r), B(s) \quad (9)$$

$$\sum_p x_{ap}^w(t^i) = x_a^w(t^i), \quad \sum_{w,p} x_{ap}^w(t^i) = \sum_w x_a^w(t^i) = x_a(t^i) \quad \forall w, a \quad (10)$$

$$\sum_p E_p^w(t^i) = E^w(t^i) \quad \forall w \quad (11)$$

In condition (9), we have created an artificial inflow links into origin node  $r$ . This is common in traffic modelling where artificial links are created from the centroids (commonly referred to as zones) to the physical origin and destination nodes (see figure 1). Observe that equation (3) is well satisfied at both origin and destination nodes  $r$  and  $s$ .



**FIGURE 1 Diagrammatic Explanation of Eqn (9)**

Non negativity conditions

$$x_{ap}^w(t^i) \geq 0 \quad [\lambda], \quad v_{ap}^w(t^i) \geq 0 \quad [\xi], \quad x_{ap}^w(t^i) \geq 0 \quad \forall i, w, p \in P_w, a \quad (12)$$

$$E_p^w(t^i) \geq 0 \quad \forall i, w, p \in P_w \quad (13)$$

$$y_{itj}^w \geq 0 \quad [\rho] \quad \forall i, j, w \quad (14)$$

Boundary conditions

$$E_p^w(t^0) = 0 \quad \forall w, p \in P_w \quad (15)$$

$$x_{ap}^w(t^0) = 0 \quad \forall w, p \in P_w, a \quad (16)$$

Flow propagation constraint

$$x_{ap}^w(t^i) = \sum_{b \in \bar{p}} \left\{ x_{bp}^w[t^i + \tau_a(t^i)] - x_{bp}^w(t^i) \right\} + \left\{ E_p^w[t^i + \tau_a(t^i)] - E_p^w(t^i) \right\} \quad \forall a \in B(l); l \neq r; p, w \quad (17)$$

Relationships between state and control variables

$$\frac{d}{dt^i} x_{ap}^w(t^i) = u_{ap}^w(t^i) - v_{ap}^w(t^i) \quad \forall a, w, p \in P_w, i \quad (18)$$

$$\frac{d}{dt^i} E_p^w(t^i) = e_p^w(t^i) \quad \forall w, p \in P_w, i \quad (19)$$

The Greek letters in the square brackets are the Karush-Kuhn-Tucker (KKT) multipliers associated with the constraints.

### 3 MODEL FORMULATION

The traditional way to model road pricing is to specify the system controller's objective in the upper level and the users' objective in the lower level. This is often referred to as bi-level program. We start by defining the upper level program.

#### 3.1 System Problem (SP)

We assume that the controller's objective is to minimize the system's total travel cost: travel time cost and the cost of shifting users from one departure time interval to another. This objective is thus stated as follows:

$$\begin{aligned} \min_f \sum_i \left( \sum_w \sum_{p \in P_w} [f_p^w(t^i) \eta_p^w(t^i)] + \sum_j \sum_w [c_{t^j t^i}^w \cdot y_{t^j t^i}^w] \right) \\ \text{s.t} \\ \text{flow feasibility constraints (Eqns(1) – (7))} \\ \text{nonnegativity constraints (Eqns(12) – (14))} \end{aligned} \quad (20)$$

With the conditions in Eqn (20), the relational and definitional conditions are satisfied. The flow propagation is defined by the flow propagation conditions in Eqn (17), and the boundary conditions are hard coded.

$f_p^w(t^i)$  is the departure flow rate into route  $p$  for the  $w^{th}$  OD pair during departure time interval  $t^i$ .

For a given OD pair  $w \in W$ ,  $c_{t^j t^i}^w$  is the cost involved in shifting departure time of a user from  $t^j$  to  $t^i$  [2]. The cost-matrix  $c_{TT}^W$  is determined using the technique of reversed engineering [2]. This technique involves the use of Small's formulation of the time of travel choice problem, to determine the preferred time of departure  $t^H$  by the users given the observed departure time pattern  $t^J$  [11]. Knowing the preferred departure time  $t^H$ , one can then determine the cost involved in shifting demand from  $t^J$  to  $t^I$ .

$\eta_p^w(t^i)$  is travel time experienced over route  $p$  by users belonging to the  $w^{th}$  OD pair. Note that we have used  $\eta_p^w(t^i)$  to mean  $\eta_p^w(f_p^w(t^i))$ . Assumption: We assume that the route cost  $\eta_p^w(t^i)$  is a continuously differentiable function of the route flow  $f_p^w(t^i)$ .

For the OD pair  $w \in W$ ,  $y_{t^j t^i}^w$  is the number of users that in the observed travel pattern  $\hat{d}$  were departing during departure time interval  $t^j$ , and will be departing in the interval  $t^i$  in the optimized pattern  $d$ .

The first part of the objective minimizes system travel time cost and the second part minimizes the system cost involved in shifting departure time of a user from  $t^j$  to  $t^i$ . Note that the choice of travel time cost  $f_p^w(t^i) \eta_p^w(t^i)$  is an arbitrary choice. The system controller instead can minimize the cost of emission, noise, etcetera or any combination of the cost as deemed fit.

If we let  $L$  to be the Lagrangian, and  $\bar{f}_p^w(t^i)$ ,  $\bar{y}_{t^j t^i}^w$  (with the corresponding path cost  $\bar{\eta}_p^w(t^i)$ ) be the solution of program (20), then, for a given  $t^i$ , there exists  $(\alpha, \gamma, \delta, \zeta, \lambda, \xi, \rho)$  such that the following *KKT* conditions hold:

$$\begin{aligned} L = & \sum_i \sum_w \sum_{p \in P_w} [f_p^w(t^i) \eta_p^w(t^i)] + \sum_i \sum_j \sum_w [c_{t^j t^i}^w \cdot y_{t^j t^i}^w] + \left( \sum_{a \in A(t)} u_{ap}^w(t^i) - f_p^w(t^i) \right) \alpha \\ & + \left( \sum_{a \in B(n)} v_{ap}^w(t^i) - \sum_{a \in A(n)} u_{ap}^w(t^i) \right) \gamma + \left( d^w(t^i) - \sum_p f_p^w(t^i) \right) \delta \\ & + \left( \sum_j y_{t^j t^i} - d^w(t^i) \right) \delta + \left( \hat{d}^w(t^i) - \sum_i y_{t^j t^i} \right) \zeta + \left( \sum_i \hat{d}^w(t^i) - \sum_i d^w(t^i) \right) \varsigma \\ & - u_{ap}^w(t^i) \lambda - v_{ap}^w(t^i) \xi - y_{t^j t^i}^w \rho \end{aligned}$$



$$\frac{\partial}{\partial f_p^w(t^i)} L = \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) \right) - \alpha_p^w(t^i) - \delta^w(t^i) = 0 \quad \forall w, p \in P_w \quad (21)$$

$$\frac{\partial}{\partial u_{ap}^w(t^i)} L = \alpha_p^w(t^i) - \gamma_p^w(t^i) - \lambda_{ap}^w(t^i) = 0 \quad \forall w, p \in P_w, a \in p \quad (22)$$

$$\frac{\partial}{\partial v_{ap}^w(t^i)} L = \gamma_p^w(t^i) - \xi_{ap}^w(t^i) = 0 \quad \forall w, p \in P_w, a \in p \quad (23)$$

$$\frac{\partial}{\partial (d^w(t^i))} L = \delta^w(t^i) - \delta^w(t^i) - \zeta^w(t^i) = 0 \quad \forall w \quad (24)$$

$$\frac{\partial}{\partial y_{ij}^w(t^i)} L = c_{ij}^w + \delta^w(t^i) - \zeta^w(t^i) - \rho^w(t^i) = 0 \quad \forall w, j \quad (25)$$

$$u_{ap}^w(t^i) \lambda_{ap}^w(t^i) = v_{ap}^w(t^i) \xi_{ap}^w(t^i) = 0 \quad \forall w, p \in P_w, a \in p \quad (26)$$

$$y_{ij}^w(t^i) \rho^w(t^i) = 0 \quad \forall j, w \quad (27)$$

$$\lambda_{ap}^w(t^i), \xi_{ap}^w(t^i) \geq 0 \quad \forall w, p \in P_w, a \in p; \quad \rho^w(t^i) \geq 0 \quad \forall w \quad (28)$$

Equations (26) & (27) are complementarity conditions.

From Eqn (21)

$$\begin{aligned} \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) \right) &= \alpha_p^w(t^i) + \delta^w(t^i) \\ &= \gamma_p^w(t^i) + \lambda_{ap}^w(t^i) + \delta^w(t^i) \quad (\text{Eqn(22)}) \\ &= \xi_{ap}^w(t^i) + \lambda_{ap}^w(t^i) - c_{ij}^w + \zeta^w(t^i) + \rho^w(t^i) \quad (\text{Eqn(23)\&(25)}) \\ \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) \right) + c_{ij}^w &\geq \zeta^w(t^i) \quad (\text{Eqn(28)}) \end{aligned}$$

Thus we have

$$\boxed{\left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{ij}^w \right) \geq \zeta^w(t^i) \quad \forall w, i, j} \quad (29)$$

Again, from Eqn (21)

$$\begin{aligned} \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) \right) &= \alpha_p^w(t^i) + \delta^w(t^i) \\ &= \gamma_p^w(t^i) + \lambda_{ap}^w(t^i) + \delta^w(t^i) \quad (\text{Eqn(22)}) \\ &= \xi_{ap}^w(t^i) + \lambda_{ap}^w(t^i) + \delta^w(t^i) \quad (\text{Eqn(23)}) \\ \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{ij}^w \right) &= \xi_{ap}^w(t^i) + \lambda_{ap}^w(t^i) + \zeta^w(t^i) + \rho^w(t^i) \quad (\text{Eqn(25)}) \\ \sum_{a \in A(r)} \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{ij}^w \right) u_{ap}^w(t^i) &= \sum_{a \in A(r)} (\xi_{ap}^w(t^i) + \lambda_{ap}^w(t^i) + \zeta^w(t^i) + \rho^w(t^i)) u_{ap}^w(t^i) \\ &= \sum_{a \in A(r)} (\xi_{ap}^w(t^i) + \zeta^w(t^i) + \rho^w(t^i)) u_{ap}^w(t^i) \quad (\text{Eqn(26)}) \\ &= \sum_{a \in A(r)} (\zeta^w(t^i) + \rho^w(t^i)) u_{ap}^w(t^i) \quad (\text{Eqn(9)\&(26)}) \\ \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{ij}^w \right) \sum_{a \in A(r)} u_{ap}^w(t^i) &= (\zeta^w(t^i) + \rho^w(t^i)) \sum_{a \in A(r)} u_{ap}^w(t^i) \\ \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{ij}^w \right) \bar{f}_p^w(t^i) &= (\zeta^w(t^i) + \rho^w(t^i)) \bar{f}_p^w(t^i) \\ \sum_{p \in P_w} \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{ij}^w \right) \bar{f}_p^w(t^i) &= \sum_{p \in P_w} (\zeta^w(t^i) + \rho^w(t^i)) \bar{f}_p^w(t^i) \\ &= (\zeta^w(t^i) + \rho^w(t^i)) \sum_{p \in P_w} \bar{f}_p^w(t^i) \\ &= (\zeta^w(t^i) + \rho^w(t^i)) \bar{d}^w(t^i) \quad (\text{Eqn(4)}) \end{aligned} \quad (30)$$

but from Eqn (5),

$$\sum_j y_{t^j t^i}^w = d^w(t^i)$$

thus we have in Eqn (30)

$$\rho^w(t^i) d^w(t^i) = \sum_j \rho^w(t^i) y_{t^j t^i}^w = 0; \quad (\rho^w(t^i) y_{t^j t^i}^w = 0, \text{ see Eqn (27)})$$

Equation (30) now becomes

$$\boxed{\sum_{p \in P_w} \left( \bar{\eta}_p^w(t^i) + \bar{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\bar{\eta}_p^w(t^i)) + c_{t^j t^i}^w \right) \bar{f}_p^w(t^i) = \zeta^w(t^i) \bar{d}^w(t^i) \quad \forall w, i, j} \quad (31)$$

### 3.2 User Problem (UP)

The user problem usually formulated in the lower level of the bi-level road pricing problem is the time dependent version of the static Wardrop's equilibrium law. We define this time dependent user equilibrium (TDUE) to be the state of the traffic in which no user thinks he/she can decrease his/her generalised travel cost by unilaterally changing routes or departure time. It has been shown that this equilibrium condition can be found by solving an equivalent variational inequality (VI) problem [6, 10, 3]. Adding the the transfer cost  $c_{t^j t^i}^w$  (which users incur by departing at  $t^i$  instead of  $t^j$ ), we reformulate the VI for the DUE in [6, 5] as follows:

Given that  $F$  is the set of all feasible path flows, then

$$\text{Find } \tilde{f}_p^w(t^i) \in F \text{ such that } \sum_w \sum_i \left( \sum_{p \in P_w} (\tilde{\eta}_p^w(t^i)) (f_p^w(t^i) - \tilde{f}_p^w(t^i)) + \sum_j [c_{t^j t^i}^w \cdot z_{t^j t^i}^w] \right) \geq 0 \quad \forall f_p^w(t^i) \in F \quad (32)$$

where  $\tilde{\eta}_p^w(t^i) = \min_{p \in P_w} \{ \eta_p^w(t^i) \}$  is the cost of the shortest path traversing the  $w^{th}$  OD pair at time  $t^i$  given the traffic flow  $\tilde{f}_p^w(t^i)$  in the network  $G$ .

Again, for a given OD pair  $w \in W$ ,  $c_{t^j t^i}^w$  is the cost involved in shifting departure time of a user from  $t^j$  to  $t^i$  [2].  $z_{t^j t^i}^w$  is the number of users that prefer to depart during departure time interval  $t_j$  but are actually departing during  $t_i$ . The tilde ' $\sim$ ' indicates a fixed parameter. Note also that the transfer cost  $c_{t^j t^i}^w$  is fixed.

The variational inequality above can be written as a minimization problem

$$\begin{aligned} \min_f \sum_w \sum_i \left( \sum_{p \in P_w} [\tilde{\eta}_p^w(t^i) f_p^w(t^i)] + \sum_j [c_{t^j t^i}^w \cdot z_{t^j t^i}^w] \right) \\ \text{s.t.} \\ f_p^w(t^i) \in F \end{aligned}$$

and thus we formulate the UP as follows:

$$\begin{aligned} \min_f \sum_w \sum_i \left( \sum_{p \in P_w} [\tilde{\eta}_p^w(t^i) f_p^w(t^i)] + \sum_j [c_{t^j t^i}^w \cdot z_{t^j t^i}^w] \right) \\ \text{s.t.} \\ f_p^w(t^i) = \sum_{a \in A(r)} u_{ap}^w(t^i) \quad \forall i, w, p \in P_w, \quad [\alpha] \\ e^w(t^i) = \sum_{a \in B(s)} \sum_{p \in P_w} v_{ap}^w(t^i) \quad \forall i, w \\ \sum_{a \in A(l)} u_{ap}^w(t^i) = \sum_{a \in B(l)} v_{ap}^w(t^i) \quad \forall i, w, p \in P_w, l \quad [\gamma] \\ \sum_{p \in P_w} f_p^w(t^i) = \hat{d}^w(t^i) \quad \forall i, w \quad [\delta] \\ \sum_j z_{t^j t^i}^w = \hat{d}^w(t^i) \quad \forall i, w \quad [\delta] \end{aligned} \quad (33)$$

Note that the definitional constraints (Eqns (8) - (12)), non-negativity constraints (Eqns (12) - (13)), boundary conditions (Eqns (15) - (16)), flow propagation conditions (Eqn (17)) and relationship between state and control variables (Eqns (18) - (19)) are also applicable.

Observe from system (33) that the route travel cost  $\tilde{\eta}_p^w(t^i)$  is fixed in accord with the VI in Eqn (32). This is actually the main difference between the system problem (20) and the user problem (33).

Given that  $\tilde{f}_p^w(t^i), \tilde{z}_{t^i}^w$  solves the user problem (33), then analysing the KKT optimality conditions (for a given  $t^i$ ) as we did in subsection 3.1 yields the following results:

$$\boxed{\tilde{\eta}_p^w(t^i) + c_{t^i}^w = \xi_{ap}^w(t^i) + \lambda_{ap}^w(t^i) + \delta^w(t^i) \quad \forall p \in P_w, w \in W, i, j} \quad (34)$$

If for route  $p$ , the inflow during time  $t^i$  into  $p$  is positive, that is  $\tilde{f}_p^w(t^i) > 0$ , then from Eqn (9) it means that  $u_{bp}^w(t^i) = v_{ap}^w(t^i) > 0 \quad \forall a \in A(r), b \in B(r)$ . Consequently, the complementarity conditions in Eqn (26) force the variables  $\xi_{ap}^w(t^i)$  and  $\lambda_{ap}^w(t^i)$  in Eqn (34) to be zero. Thus we have the following

$$\boxed{\tilde{\eta}_p^w(t^i) + c_{t^i}^w = \delta^w(t^i) \quad \forall \tilde{f}_p^w(t^i) > 0, p \in P_w, w \in W, i, j} \quad (35)$$

The LHS of (35) is total equilibrated cost for traversing OD pair  $w \in W$  using route  $p \in P_w$  for users departing origin  $r$  towards destination  $s$  during time  $t^i$ . Observe that the RHS is a variable that does not depend on  $p$ .

Recall that  $\tilde{\eta}_p^w(t^i)$  is the travel cost on route  $p \in P_w$  and  $c_{t^i}^w$  is the cost a user incurs for departing during  $t^i$  instead of  $t^j$ .

Interpretation: At equilibrium, the travel costs on all used routes for a given OD pair  $w \in W$  are the same and equal to  $\delta^w(t^i)$  for all users departing during time  $t^i$ .

The following holds in general due to Eqn (28):

$$\boxed{\tilde{\eta}_p^w(t^i) + c_{t^i}^w \geq \delta^w(t^i) \quad \forall w \in W, i, j} \quad (36)$$

Interpretation: At equilibrium, the travel cost for all users departing origin during time  $t^i$  towards the same destination for a given OD pair  $w \in W$  is greater or equal to  $\delta^w(t^i)$  irrespective of the route they use. This means that at equilibrium,  $\delta^w(t^i)$  must be the least travel cost between the OD pair  $w \in W$  for users departing during time  $t^i$ . Recall that from (35)  $\delta^w(t^i)$  is the travel cost on all used path. We thus state the following: at equilibrium, the travel costs on all used paths for a given OD pair are the same and less or equal to those on unused paths (Wardrop's first principle).

With the above interpretation, we therefore conclude that any path flow  $\tilde{f}_{P_w}^w(t^T)$  vector that solves system (33) is a user equilibrium flow. The proof follows from the KKT analysis and argument given above.

Furthermore, following the same lines of argument that led to Eqn (31), we arrive at the following:

$$\boxed{\sum_{p \in P_w} (\tilde{\eta}_p^w(t^i) + c_{t^i}^w) \tilde{f}_p^w(t^i) = \delta^w(t^i) \hat{d}^w(t^i) \quad \forall w \in W, i, j} \quad (37)$$

Eqn (37) is the network cost balance equation.

### 3.3 The First-Best time dependent Pricing Scheme

Now compare Eqn (29) with (36) and (31) with (37) and observe that the difference between them is the quantity

$\left( \tilde{f}_p^w(t^i) \frac{d}{df_p^w(t^i)} (\tilde{\eta}_p^w(t^i)) \right)$  in the analysis of the SP. Therefore by adding the term

$$\left( f_p^w(t^i) \frac{d}{df_p^w(t^i)} (\eta_p^w(t^i)) \right) \Big|_{f_p^w(t^i) = \tilde{f}_p^w(t^i)}$$

to the path travel cost  $\tilde{\eta}_p^w(t^i) \quad \forall p \in P_w, w \in W$ , the first order optimality conditions of user problem will exactly be the same as those of system problem. This means that any flow pattern that solves the system problem will also solve the user problem (i.e.  $\tilde{f}_p^w(t^i) = \tilde{f}_p^w(t^i) \quad \forall p \in P_w, w \in W$ ).

If we denote by  $\theta_p^w(t^i)$  the optimal toll to be paid on route  $p \in P_w$  when departing origin  $r$  during time  $t^i$  towards destination  $s$ , then the first-best optimal toll can be given by

$$\bar{\theta}_p^w(t^i) = \left( f_p^w(t^i) \frac{d}{df_p^w(t^i)} (\eta_p^w(t^i)) \right) \Big|_{f_p^w(t^i) = \tilde{f}_p^w(t^i)} \quad (38)$$

where  $\bar{f}_p^w(t^i)$  is the solution of the SP.

Interpretation: The first term of the toll  $\bar{\theta}_p^w(t^i)$  is the additional travel cost imposed on all the existing users of route  $p \in P_w$  by an additional user on route  $p \in P_w$ , all departing from the same origin at the same departure interval  $t^i$  and heading towards the same destination. Therefore, by adding the toll  $\bar{\theta}_p^w(t^i)$  to the cost of travel on route  $p \in P_w$  for users departing during  $t^i$ , we now ensure that all users, before embarking on a trip, take into account the cost they incur and impose on other travellers by departing at the chosen time  $t^i$ . It turns out that  $\bar{\theta}_p^w(t^i)$  as given in Eqn (38) is not the only possible toll that can achieve the system optimal flow  $\bar{f}_p^w(t^i)$ , in fact there are infinite number of toll vectors that can achieve this optimal flow, thus we state the following:

**Theorem 1:** For all departure times  $t^i$ , any route toll  $\theta_p^w(t^i)$ ,  $p \in P_w$  satisfying the following linear conditions will also result in an optimal route flow pattern  $\bar{f}_p^w(t^i)$ :

$$\boxed{\begin{aligned} (\bar{\eta}_p^w(t^i) + c_{ij}^w + \theta_p^w(t^i)) &\geq \delta^w(t^i) & \forall w \in W, j \\ \sum_{p \in P_w} (\bar{\eta}_p^w(t^i) + c_{ij}^w + \theta_p^w(t^i)) \bar{f}_p^w(t^i) &= \delta^w(t^i) \bar{d}^w(t^i) & \forall w \in W, j \end{aligned}} \quad (39)$$

where  $\delta^w(t^i)$  is a free variable, and  $\bar{d}^w(t^i)$  is the optimal demand of users departing origin  $r$  toward destination  $s$  during time  $t^i$ .

**Proof:** The proof simply follows from the KKT conditions of the SP and UP and the argument given earlier in this subsection (similar to the ones given in [13] for static case)  $\square$ .

Note that with Eqn (39), one can easily define secondary objectives on the path tolls, for example, fixing the total toll collected, minimizing the maximum route toll over all routes, etcetera.

### 3.4 Algorithm to generate first-best route-based tolls

Step 1: Solve the system problem (20) to determine the optimal route flow  $\bar{f}_p^w(t^i)$  for all ODs and all routes

Step 2: With the optimal route flows  $\bar{f}_p^w(t^i)$ , solve the linear system (39) (with or without secondary objective) to determine a route-based toll pattern  $\theta_p^w(t^i)$  that induces the optimal route flow pattern  $\bar{f}$

Step 3: Stop

### 3.5 The Second-Best time dependent Pricing Scheme

Here we define the second-best scheme to mean a tolling scheme where tolls are not allowed on some paths. This requirement may just be for a given time interval, thus for a given origin-destination pair  $w$ , a path may be required to have a zero toll during the interval  $t^i$  and may assume a positive toll during  $t^j$  where  $i \neq j$ . If we denote by  $Y^w(t^i)$  the set of all non-tollable paths during departure interval  $t^i$  for the  $w^{th}$  OD pair, then it is required that

$$\theta_p^w(t^i) = 0 \forall p \in Y^w(t^i) \quad (40)$$

If condition (40) is required, then one only need to add this toll constraints to system (39) and solve to see if one can still achieve the optimal results  $\bar{f}_p^w(t^i)$  from system (20). If there is no feasible path toll pattern  $\theta_p^w(t^i)$  for the optimal path flow pattern  $\bar{f}_p^w(t^i)$ , then with the optimal flow pattern  $\bar{f}_p^w(t^i)$  as a starting point, one needs to solve system (20) together with addition conditions in system (39) and of course the toll constraints of system (40) as given below:

$$\begin{aligned} \min_f \sum_i &\left( \sum_w \sum_{p \in P_w} [f_p^w(t^i) \eta_p^w(t^i)] + \sum_j \sum_w [c_{ij}^w \cdot y_{ij}^w] \right) \\ & \text{s.t.} \\ & \text{flow feasibility constraints} \\ & \text{nonnegativity constraints} \\ & \left( \eta_p^w(t^i) + c_{ij}^w + \theta_p^w(t^i) \right) \geq \delta^w(t^i) \quad \forall w \in W \\ & \sum_{p \in P_w} \left( \eta_p^w(t^i) + c_{ij}^w + \theta_p^w(t^i) \right) f_p^w(t^i) = \delta^w(t^i) d^w(t^i) \quad \forall w \in W \\ & \theta_p^w(t^i) = 0 \forall p \in Y^w(t^i) \quad \forall w \in W \end{aligned} \quad (41)$$

The third and the fourth conditions ensure that the generated feasible flow is in user equilibrium.

### 3.6 Algorithm to generate second-best route-based tolls

Step 1: Solve the system problem (20) to determine the optimal route flow  $\bar{f}_p^w(t^i)$  for all ODs and all routes

Step 2: With the optimal route flows  $\bar{f}_p^w(t^i)$ , solve the linear system (39) with the toll constraint of Eqn (40) to determine a route-based toll pattern  $\theta_p^w(t^i)$  that induces the optimal route flow pattern  $\bar{f}$ . If such toll pattern exists, GOTO step 4, otherwise, GOTO step 3

Step 3: With the optimal flow pattern  $\bar{f}_p^w(t^i)$  as a starting point, solve system (41) to determine the second-best path-based tolls  $\theta_p^w(t^i)$

Step 4: Stop

## 4 NUMERICAL EXAMPLE

We will demonstrate our model using a five-node network with eight links (see figure 1). In our model and derivations, we have focused on travel time cost, recall that the choice of travel time cost  $f_p^w(t^i) \eta_p^w(t^i)$  is an arbitrary choice. A decision maker may as well choose to optimize emission, noise, safety etcetera. In this specific example, we have chosen to optimize the entire travel cost and transfer costs over a given time horizon  $T$ . We suppose that the time window  $T$  is divided into 3 discrete time intervals  $t^i$  with  $i = 1, 2, 3$ . Recall again that the  $t^i$ 's correspond to the departure time intervals. We further suppose undifferentiated users, model with different user classes is straightforward. Next we give the link attributes and input for the model.

### 4.1 Five-Node Network Example

#### 4.1.1 Link Attributes and Input

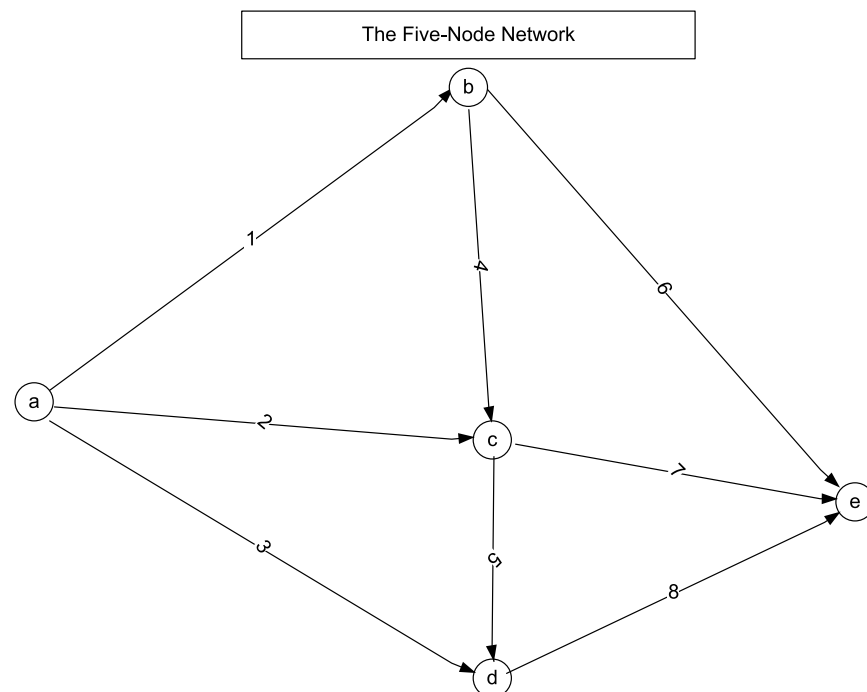


FIGURE 2 The five-node network with eight links.

TABLE 2 Network Attributes

**2a. Link Attributes**      **Vehicle Class = Private cars**

Link	Length(km)	Free Speed(km/hr)	Capacity
1	10	100	400
2	7	70	300
3	10.5	100	350
4	5	70	200
5	4	70	250
6	10	90	250
7	5	80	250
8	8.5	90	300

**2b. Transfer cost ( $C_{TT}$ )**

	$t^1$	$t^2$	$t^3$
$t^1$	0.00	0.43	0.37
$t^2$	0.43	0.00	0.80
$t^3$	0.37	0.80	0.00

**2c. Input OD demand (counted)**

$d(t^1)$	400
$d(t^2)$	1000
$d(t^3)$	100

Table 2a gives the link characteristics for the eight links. 2b gives the transfer costs  $c_{t^j t^i}$  involved in shifting departure time of a user from  $t^j$  (observed or counted) to  $t^i$  (optimized) [2]. Recall that the cost-matrix  $c_{TT}$  can be determined using the technique of reversed engineering as described in [2]. Table 2c gives the observed daily traffic pattern for the example network and for the three discrete times  $t^1$ ,  $t^2$  and  $t^3$ .

We use the so called *Bureau for Public Roads (BPR) function*  $\beta T_a^{ff} \left( 1 + \varphi \left( \frac{v_a(t^i)}{\hat{C}_a} \right)^\phi \right)$  to define the link travel time cost during time  $t^i$ , where

$T_a^{ff}$  - free flow travel time on link  $a$ ,

$v_a(t^i)$  - total flow on link  $a$  during  $t^i$ , note that we have allowed  $v_a(t^i)$  to constitute the remaining link inflows during  $t^{i-1}$ , where  $v_a(t^0) = 0$ . This fraction (remaining inflow) is based on the travel time on the chain of links preceding link  $a$  and the length of the departure times.

$\hat{C}_a$  - practical capacity of link  $a$ , and

$\varphi$  and  $\phi$  - BPR scaling parameters.

We set  $\varphi = 0.15$ ,  $\phi = 4$  and  $\beta$  (value of time - VOT) = 0.1671667 EUR/minute. The value of time (VOT) used is as stated in [1].

$$\eta_p^w(t^i) = \sum_a x_{ap}^w(t^i).$$

## 4.2 Results

As a reference point, we solve the user problem as described in subsection 3.2. This describes the traffic situation without tolling, and the results are given in table 3 below.

**TABLE 3 Observed Traffic Scenario for Three Discrete Time Intervals**

Tolls and costs are in Euro (€)  
Observed table

$t^1$						Corresponding
Paths [p]	Path flows [ $f_p$ ]	Path tolls [ $\theta_p$ ]	Path Cost [ $\eta_p$ ]	System Cost [ $f_p \eta_p$ ]	Links	link flows ( $v_a$ )
a--b--e [i]	0	0.00	2.12	0.00	1	0
a--b--c--e [ii]	0	0.00	2.56	0.00	2	306
a--b--c--d--e [iii]	0	0.00	3.24	0.00	3	94
a--c--e [iv]	306	0.00	2.00	612.34	4	0
a--c--d--e [v]	0	0.00	2.69	0.00	5	0
a--d--e [vi]	94	0.00	2.00	188.72	6	0
<b>Demand <math>d(t^1)</math></b>	<b>400</b>			<b>801.06</b>	<b>7</b>	<b>306</b>
					<b>8</b>	<b>94</b>

$t^2$						Corresponding
Paths [p]	Path flows [ $f_p$ ]	Path tolls [ $\theta_p$ ]	Path Cost [ $\eta_p$ ]	System Cost [ $f_p \eta_p$ ]	Links	link flows ( $v_a$ )
a--b--e [i]	284	0.00	2.43	690.40	1	284
a--b--c--e [ii]	0	0.00	2.83	0.00	2	365
a--b--c--d--e [iii]	0	0.00	3.55	0.00	3	351
a--c--e [iv]	365	0.00	2.43	889.15	4	0
a--c--d--e [v]	0	0.00	3.15	0.00	5	0
a--d--e [vi]	351	0.00	2.43	853.47	6	284
<b>Demand <math>d(t^2)</math></b>	<b>1,000</b>			<b>2,433.01</b>	<b>7</b>	<b>365</b>
					<b>8</b>	<b>351</b>

$t^3$						Corresponding
Paths [p]	Path flows [ $f_p$ ]	Path tolls [ $\theta_p$ ]	Path Cost [ $\eta_p$ ]	System Cost [ $f_p \eta_p$ ]	Links	link flows ( $v_a$ )
a--b--e [i]	0	0.00	2.12	0.00	1	0
a--b--c--e [ii]	0	0.00	2.35	0.00	2	100
a--b--c--d--e [iii]	0	0.00	3.24	0.00	3	0
a--c--e [iv]	100	0.00	1.64	163.55	4	0
a--c--d--e [v]	0	0.00	2.53	0.00	5	0
a--d--e [vi]	0	0.00	2.00	0.00	6	0
<b>Demand <math>d(t^3)</math></b>	<b>100</b>			<b>163.55</b>	<b>7</b>	<b>100</b>
					<b>8</b>	<b>0</b>

Transfers ( $Zt^i$ )	$t^1$	$t^2$	$t^3$
$t^1$	400	0	0
$t^2$	0	1,000	0
$t^3$	0	0	100

Total transfer cost to:	
$t^1$	0.00
$t^2$	0.00
$t^3$	0.00
<b>Total</b>	<b>0.00</b>

Observe that with the transfer costs given in table 2b, the flows in table 3 are in user equilibrium. As mentioned earlier, our aim is to efficiently distribute the traffic flows across the departure time interval using tolls.

Given that all paths (links) can be tolled, we solve the system problem (system (20)) for optimal path flows (hence the link flows)  $\bar{f}_p^w(t^i)$  that minimize the entire network cost. As described in subsection 3.4, using the optimal path flows we search for a path toll pattern  $\theta_p^w(t^i)$  (for all paths) that will induce the optimal path flows using the linear system as given in (39) (see table 4). Note that we have only one OD pair, and as such  $w$  corresponds to the pair  $ae$ . In the following tables, we give the results of the optimal time dependent tolls for the time dependent traffic assignment.

**TABLE 4 Optimized Traffic Scenario for Three Discrete Time Intervals**

**Optimized table**

$t^1$							Corresponding
Paths [p]	Path flows [f <sub>p</sub> ]	Path tolls [θ <sub>p</sub> ]	Path Cost [η <sub>p</sub> ]	System Cost [f <sub>p</sub> η <sub>p</sub> ]	Links		link flows (v <sub>a</sub> )
a--b--e [i]	150	0.45	2.59	388.79	1		150
a--b--c--e [ii]	0	1.01	3.42	0.00	2		227
a--b--c--d--e [iii]	0	0.16	3.42	0.00	3		186
a--c--e [iv]	227	0.85	2.59	588.88	4		0
a--c--d--e [v]	0	0.00	2.59	0.00	5		0
a--d--e [vi]	186	0.56	2.59	481.66	6		150
<b>Demand d(t<sup>1</sup>)</b>	<b>563</b>			<b>1,459.33</b>	<b>7</b>		<b>227</b>
					<b>8</b>		<b>186</b>

$t^2$							Corresponding
Paths [p]	Path flows [f <sub>p</sub> ]	Path tolls [θ <sub>p</sub> ]	Path Cost [η <sub>p</sub> ]	System Cost [f <sub>p</sub> η <sub>p</sub> ]	Links		link flows (v <sub>a</sub> )
a--b--e [i]	211	0.37	2.59	547.85	1		211
a--b--c--e [ii]	0	0.12	2.59	0.00	2		256
a--b--c--d--e [iii]	0	0.00	3.33	0.00	3		256
a--c--e [iv]	256	0.76	2.59	664.82	4		0
a--c--d--e [v]	0	0.64	3.33	0.00	5		0
a--d--e [vi]	256	0.47	2.59	663.14	6		211
<b>Demand d(t<sup>2</sup>)</b>	<b>723</b>			<b>1,875.81</b>	<b>7</b>		<b>256</b>
					<b>8</b>		<b>256</b>

$t^3$							Corresponding
Paths [p]	Path flows [f <sub>p</sub> ]	Path tolls [θ <sub>p</sub> ]	Path Cost [η <sub>p</sub> ]	System Cost [f <sub>p</sub> η <sub>p</sub> ]	Links		link flows (v <sub>a</sub> )
a--b--e [i]	0	0.48	2.59	0.00	1		0
a--b--c--e [ii]	0	0.87	3.24	0.00	2		173
a--b--c--d--e [iii]	0	0.00	3.24	0.00	3		41
a--c--e [iv]	173	0.92	2.59	448.28	4		0
a--c--d--e [v]	0	0.05	2.60	0.00	5		0
a--d--e [vi]	41	0.59	2.59	107.05	6		0
<b>Demand d(t<sup>3</sup>)</b>	<b>214</b>			<b>555.33</b>	<b>7</b>		<b>173</b>
					<b>8</b>		<b>41</b>

Transfers (yt <sup>t</sup> )	$t^1$	$t^2$	$t^3$
$t^1$	400	0	0
$t^2$	163	723	114
$t^3$	0	0	100

Total transfer cost to:	
$t^1$	70.00
$t^2$	0.00
$t^3$	91.00
<b>Total</b>	<b>161.01</b>

**TABLE 5 Summary Table**

<b>Summary</b>	
<b>Total demand observed</b>	1,500.00
<b>Total demand optimized</b>	1,500.00
<b>Total system cost observed</b>	3,397.62
<b>Total system cost optimized</b>	3,109.17
<b>Cost reduction</b>	288.46
<b>% cost reduction</b>	8.49



Observe from tables 3 and 4 that the traffic flows obey Wardrop's equilibrium, where the cost on all used paths are the same and smaller than the costs on the unused paths, confirming Eqn (36). Observe again from both tables that traffic flows obey Wardrop's equilibrium in and over the time horizon  $T$ . This in particular ensures that no user will be better off by switching routes within a given departure time interval or by changing his/her departure time. With reference to our model in Eqn (39), it means that the least travel path cost is given by  $\delta^w(t^i) = 2.59$ . The last two adjacent tables of table 4 give the flow transfers and the cost of the transfers.

Though we are not optimizing over the tolls, it is interesting to see that the path tolls required to achieve the desired path flows are very small for all paths. This means that it may even be possible to further reduce these path tolls if we further optimize over the path toll. It is important to note that the (path) toll patterns as given in table 4 are in general not unique. In fact, there exist infinite (path) toll patterns that can achieve the desired (path) flow pattern (see *theorem 1*).

Note that in table 4, the figures under the *path cost* (column 4) have in them the path travel time cost and the path toll cost, therefore the *system cost* (column 5) has in it, the route toll costs and the route travel time costs. Since we are not optimizing over the tolls, the optimization of the total system cost Eqn (20) as described in section 3.1 does not include the path tolls in the path travel cost  $\eta_p(t^i)$ . Consequently, the total system cost as summarized in the summary table does not include the route toll costs. Therefore, to get the values in the summary table, one has to do the following

$$Total\ system\ cost = \sum_i \left( \sum_p [\bar{f}_p(t^i) \bar{\eta}_p(t^i)] + \sum_j [c_{t^i j} \cdot \bar{y}_{t^i j}] \right) - \sum_i \sum_p \bar{f}_p(t^i) \bar{\theta}_p(t^i)$$

The bar '-' signifies optimal results.

So, by charging a user who departs origin  $a$  at time  $t^i$  travelling towards destination  $e$  over route  $p$  a value equal to  $\bar{\theta}_p(t^i)$  as given in table 4, we can assure that the system will tend to a Wardrop's equilibrium which coincides with the system optimal flow pattern  $\bar{f}_p^w(t^i)$ . The summary table 5 shows approximately 9% cost reduction (with respect to no-toll scenario) for this simple network.

Recall from subsection 3.5 that a second-best requires some paths not to be tolled at a given departure time interval, so (for example) if we are required to keep path  $v$  toll free during  $t^1$  and path  $iii$  toll free during  $t^2$  and  $t^3$ , it means then that we can still achieve our system optimal flow  $\bar{f}_p^w(t^i)$  with the tolls in table 4. On the other hand, with a different requirement on tolls, it may be that the optimal system flow  $\bar{f}_p^w(t^i)$  can no longer be achieved (see subsection 3.5).

## 5 CONCLUSION

Utilizing the basic features of dynamic traffic models, we have extended the static traffic assignment tolling model to include those tolls that change with time of the day. Using transformed time dependent user equilibrium (TDUE) in the form of variational inequality (VI), we have analytically derived a flexible route-based optimal tolling scheme that can be used to efficiently distribute traffic in and over time in a transportation network. The time dependent tolling scheme also considers users' preferred departure times and transfer costs.

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