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NUMERICAL DIFFUSION IN TRAFFIC FLOW SIMULATIONS

Accuracy analysis based on the modified equation method

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ABSTRACT

Traffic flow simulation tools only solve the model equations accurately to a certain extend. For several reasons it is important to understand the causes of inaccuracies. We focus on inaccuracies caused by numerical methods, in particular numerical diffusion which is analyzed using the modified equation method. We conclude that and explain why application of the Lagrangian formulation to a traffic flow model leads to less numerical diffusion than the traditional Eulerian formulation.

KEYWORDS

Traffic flow, simulation, Lagrangian coordinates, accuracy, modified equation method

INTRODUCTION

Traffic flow models and simulation tools are often used for traffic state estimation and prediction. Traffic flow models such as the kinematic wave model, 'higher order' models and car-following models are more or less accurate representations of reality. In a simulation tool the model equations are solved using numerical methods, again more or less accurately. The inaccuracies in the step from model equation to simulation result are mostly caused by the applied numerical methods. It is important to understand numerical inaccuracies for several reasons: 1) to understand and interpret simulation results 2) to develop better numerical methods 3) to prevent trying to improve the model or its parameters, while the inaccuracy comes from the numerical method.

In this article we focus on one kind of numerical inaccuracies, namely numerical diffusion. This causes numerical results to be smoother than the analytical solution, see Figure 1. Therefore, even if the underlying model would be perfect, and no other numerical inaccuracies are introduced the simulation will show inaccuracies. Analysis of numerical diffusion is important for any application of traffic flow simulations. For example, as a result of numerical diffusion the simulation can predict free flow while the analytical solution would show congestion at the location of an on ramp. In networks this can have a great impact on spill back. The 'amount' of the numerical diffusion depends on many factors: the discretization characteristics (road segment size, time step size and their ratio), spatial integration method, time stepping method and coordinate system. It is important to note that even though it is sometimes argued that there is too little diffusion in some traffic flow models such as the kinematic wave model, it should not be introduced by the numerical method. If one wants to introduce diffusion, this should be done by adjusting the model equations. Any numerical diffusion depends on the actual settings of the parameters of the numerical method and is, therefore, hard to control.

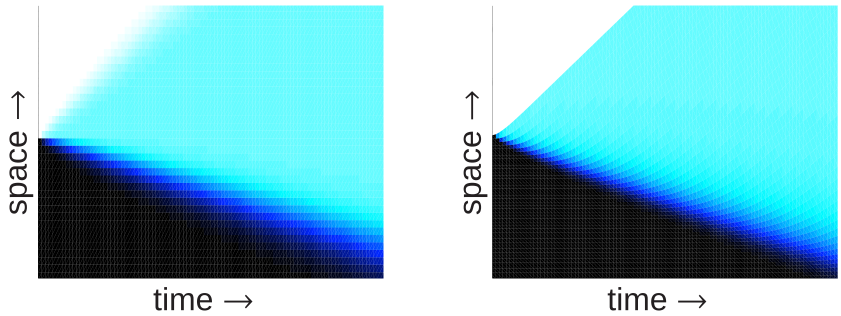

Figure 1: Density plots in t-x plane showing simulations results for a simple test problem representing a traffic signal turning green at $t = 0$. The sharp edge between the low-density region downstream of the traffic sign (white) and the region at capacity (light blue) is clearly visible in the results based on the Lagrangian formulation (right). The results based on the Eulerian formulation (left) are more diffusive.

The main contribution is an analysis of numerical diffusion caused by the coordinate system and the spatial integration method. The difference in numerical diffusion between methods based on the Eulerian and those based on the Lagrangian formulation was observed by Leclercq et al. (2007) and Van Wageningen-Kessels et al. (2010). We apply the modified equation method introduced in Warming & Hyett (1974) to explain the previously observed difference. The method is applied to the kinematic wave model Lighthill & Witham (1955), Richards (1956). However, it can also be applied for analysis of numerical diffusion in other continuum models such as multi-class models and higher-order models.

NUMERICAL DIFFUSION TERM

The kinematic wave traffic flow model is based on the conservation equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0, \tag{1}$$

with ρ the vehicle density [vehicles/m], q the vehicle flow [vehicles/s] and t and x the time and space coordinate respectively. Numerical methods are based on spatial and time discretization methods. Spatial discretization of equation (1) is usually based on the Godunov scheme, Lebacque (1996). We apply the modified equation method (which will be discussed in more detail in the full paper) to the semi-discretized equation. This results in rewriting the conservation equation (1) as an advection diffusion equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = \left| \frac{\mathrm{d}q}{\mathrm{d}\rho} \right| \frac{\Delta x}{2} \frac{\partial^2 \rho}{\partial x^2}, \tag{2}$$

with Δx the grid cell size. In a simulation this equation (2) is solved more accurately than the original conservation equation (1). The right-hand side of (2) is a diffusion term with diffusion coefficient $\gamma = |\mathrm{d}q/\mathrm{d}\rho|\Delta x/2$. We use the diffusion coefficient γ to quantify the numerical diffusion.

DIFFUSION AND THE COORDINATE SYSTEM

The analysis above is based on the conservation equation in the traditional Eulerian coordinates. We apply the same method to the conservation equation with a moving coordinate system. Again we find that an advection-diffusion equation is solved more accurately than the conservation equation. The diffusion coefficient now depends on the grid cell size and the coordinate velocity. In the Lagrangian formulation the coordinate velocity is equal to the vehicle speed, in the Eulerian formulation coordinates are fixed in space. Also any other coordinate velocity could be chosen. In the Lagrangian formulation the grid cell size depends on both the the current density and the vehicle group size Δn that is used in the discretization $\Delta x(x, t) = \Delta n/\rho(x, t)$, that is: the grid cell size denotes the length of road taken by one vehicle group.

In Figure 2 the diffusion coefficient and its dependence on the traffic state, the coordinate system and the grid cell size is shown. For most values of the density, the Lagrangian formulation introduces less diffusion than the Eulerian formulation. However, for densities just above critical the vehicle group size should be taken not too large in order to have less diffusion with the Lagrangian formulation than with the Eulerian formulation. It should be noted that very large vehicle group sizes of $\Delta n = 40$ result in a low resolution and it is not surprising that the diffusion coefficient is large.

CONCLUSION

Numerical methods are used in traffic flow simulation tools to solve the model equations. For several reasons it is important to know how accurate this process is. In this article we focus on one aspect of accuracy: numerical diffusion. It makes the simulation results of a traffic flow model more smooth and might therefore give unrealistic results, irrespective of the quality of the underlying model. Numerical diffusion is analyzed by applying the modified equation method. It shows that the 'amount' of the numerical diffusion depends on the wave velocity, the grid cell size and the coordinate velocity. This explains why simulations based on the Lagrangian formulation of the conservation equation show less numerical diffusion than simulations based on the Eulerian formulation. In the full paper we will describe the numerical methods and the

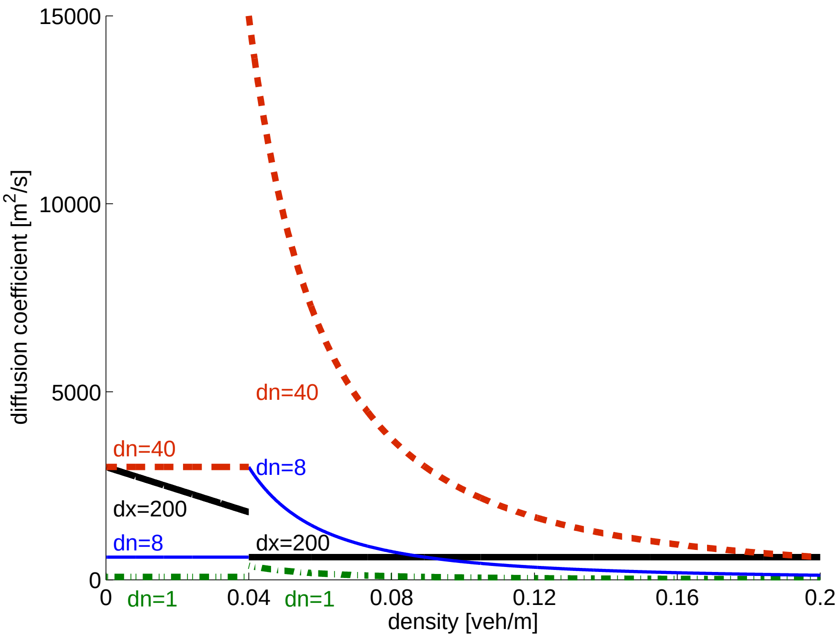

Figure 2: Diffusion coefficient as function of the density, with the Eulerian formulation ($\Delta x = 200$ m) and the Lagrangian formulation ($\Delta n = 40$ vehicles (corresponds to $\Delta x = 200$ m at jam density $\rho_{\max} = 0.2$ vehicles/m), $\Delta n = 8$ vehicles (corresponds to $\Delta x = 200$ m at critical density $\rho_{\text{crit}} = 0.04$ vehicles/m) and $\Delta n = 1$ vehicle. The Smulders fundamental diagram is used with maximum and critical speed $v_{\max} = 30$ m/s and $v_{\text{crit}} = 24$ m/s respectively.

modified equation method in more detail. Furthermore, the analysis will be extended to other coordinate velocities and to other fundamental diagrams.

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